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Nonsmooth Analysis and Approximation

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1. INTRODUCTION

Initially, optimization and the theory of best approximation developed independently. However, in the 1960s, with the appearance of convex analysis, it was realized that best approximation problems can be regarded as special problems of optimization. This led to a systematic effort to obtain approximation results as special cases of more general theorems of optimization theory. This parallel treatment is presented in the monographs of Holmes [14] and Laurent [17], which illustrate that there is a strong interaction between approximation theory and what is known by now as "nonsmooth analysis."

This paper develops along these lines and concentrates on problems of stability (sensitivity) and stochastic approximation.

In the study of stability our main tool is the so called Kuratowski–Mosco convergence of sets and the corresponding τ-convergence of proper functions. So we perturb the data determining the f-best approximations and the f-farthest points and we examine how the sets of these points vary. Such sensitivity analysis is, among other things, very important in designing efficient numerical algorithms. Additional results in this direction were recently obtained by the authors in [20].

In stochastic approximation, which is studied in Section 4, we allow both the set and the function to depend measurably on a parameter ω and we examine the dependence on ω of the various notions of approximation theory. We also study the approximation problem in which the function is the integral functional determined by $f(\cdot, \cdot)$. In all these our main tools are the theory of normal integrands of Rockafellar [23, 24] and the theory of measurable multifunctions.

Finally in Section 5 we have gathered some general results which illustrate the strong interaction between approximation theory and nonlinear analysis.

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2. PRELIMINARIES

Let (Ω, Σ) be a measurable space and X a Polish space. Let $F: \Omega \to 2^X \setminus \{\phi\}$ be a multifunction (set-valued function) with closed values. Then the following statements are equivalent:

(i) $F^-(U) = \{\omega \in \Omega: F(\omega) \cap U \neq \phi\} \in \Sigma$ for all $U \subseteq X$ open,

(ii) $\omega \to d_{F(\omega)}(x) = \inf_{z \in F(\omega)} \|x - z\|$ is measurable for all $x \in X$,

(iii) there exist measurable functions $f_n: \Omega \to X$ s.t.

$$F(\omega) = \mathrm{cl}\{f_n(\omega)\}_{n \geqslant 1} \quad \text{for all} \quad \omega \in \Omega \quad \text{(Castaing's representation).}$$

A multifunction satisfying any of the above statements is said to be measurable. If there exists a complete, σ-finite measure $\mu(\cdot)$ on Σ, (i) $\to$ (iii) are all equivalent to

(iv) $\mathrm{Gr}\, F = \{(\omega, x) \in \Omega \times X: x \in F(\omega)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X.

From now on assume that (Ω, Σ, μ) is a complete probability space and X a separable Banach sspace. By X^* we will denote its topological dual. We will use the notations

$$P_{f(c)}(X) = \{A \subseteq X: \text{nonempty, closed, (convex)}\},$$

$$P_{(w)k(c)}(X) = \{A \subseteq X: \text{nonempty, } (w\text{-})\text{compact, (convex)}\},$$

where w denotes the weak topology on X. If $A \subseteq X$ we will denote by $\sigma_A(\cdot)$ the support function of A, i.e., for all $x^* \in X^*$ $\sigma_A(x^*) = \sup_{x \in A}(x^*, x)$.

Consider the set $S^1_F = \{f(\cdot) \in L^1_X(\Omega): f(\omega) \in F(\omega)\ \mu\text{-a.e.}\}$, i.e., S^1_F contains all selectors of $F(\cdot)$ belonging to the Lebesgue–Bochner space $L^1_X(\Omega)$. Clearly S^1_F is a closed (maybe empty) subset of $L^1_X(\Omega)$. It is nonempty if and only if $\inf_{x \in F(\omega)} \|x\| \in L^1_+(\Omega)$. We will say that $F(\cdot)$ is integrably bounded if and only if $\sup_{x \in F(\omega)} \|x\| = |F(\omega)| \in L^1_+(\Omega)$. Using S^1_F we can define an integral for $F(\cdot)$,

$$\int_\Omega F(\omega)\, d\mu(\omega) = \left\{\int_\Omega f(\omega)\, d\mu(\omega): f(\cdot) \in S^1_F\right\},$$

where $\int_\Omega f(\omega)\, d\mu(\omega)$ is the usual Bochner integral. This set-valued integral is known as Aumann's integral. For more details on measurable multifunctions and their integral we refer to Castaing–Valadier [4], Himmelberg [10], and Rockafellar [23].

Now let us pass to normal integrands. These were introduced and studied by Rockafellar [23, 24], as the appropriate generalization to accommodate the needs of optimization and optimal control, of the

Caratheodory integrands from the calculus of variations. So assume that $f: \Omega \times X \to \bar{\mathbb{R}}$ is a proper integrand (i.e., $f(\cdot, \cdot)$ takes values in $(-\infty, +\infty]$ and $f \neq +\infty$). We say that $f(\cdot, \cdot)$ is a normal integrand if and only if $\omega \to \operatorname{epi} f(\omega, \cdot) = \{(x, \lambda) \in X \times \mathbb{R}: f(\omega, x) \leqslant \lambda\}$ is closed valued and measurable. A straightforward application of Von Neumann's projection theorem tells us that the above definition is equivalent to saying that $f(\cdot, \cdot)$ is $\Sigma \times B(X)$-measurable and $f(\omega, \cdot)$ is l.s.c. for all $\omega \in \Omega$. Recall that normality is preserved by the Fenchel transform. For more details the reader can look at the excellent survey paper of Rockafellar [23].

As we already mentioned in the Introduction, in the next section we will be using the Kuratowski–Mosco convergence of sets and the corresponding τ-convergence of proper functions. Very briefly we will recall those notions. Let $\{A_n, A\}_{n \geqslant 1} \subseteq 2^X$ and set $w\text{-}\overline{\lim}_{n \to \infty} A_n = \{x \in X\colon x = w\text{-}\lim_{k \to \infty} x_k, x_k \in A_{n_k}, k \geqslant 1\}$ and $s\text{-}\underline{\lim}_{n \to \infty} A_n = \{x \in X\colon x = s\text{-}\lim_{n \to \infty} x_n, x_n \in A_n, n \geqslant 1\}$. We say that the A_n's converge to A in the Kuratowski–Mosco sense (denoted by $A_n \to^{\mathrm{K\text{-}M}} A$) if and only if $w\text{-}\overline{\lim}\, A_n = A = s\text{-}\underline{\lim}\, A_n$. If $\{f_n, f\}_{n \geqslant 1} \subseteq \bar{\mathbb{R}}^X$ are proper functions then $\{f_n\}_{n \geqslant 1}$ τ-converges to f (denoted by $f_n \to^{\tau} f$) if and only if $\operatorname{epi} f_n \to^{\mathrm{K\text{-}M}} \operatorname{epi} f$. For more details we refer to Mosco [19] and Salinetti–Wets [25].

A last piece of terminology. If $f \in \bar{\mathbb{R}}^X$ is proper, by $\operatorname{dom} f$ we denote the effective domain of $f(\cdot)$, i.e., $\operatorname{dom} f = \{x \in X: f(x) < +\infty\}$. Moreover, all $\bar{\mathbb{R}}$-valued functions will be assumed to be proper.

3. Stability Results

If $f \in \bar{\mathbb{R}}^X$ is a proper function and $A \subseteq X$ nonempty, then we set $f_A(x) = \inf_{y \in A} f(x - y)$ and $P_{f,A}(x) = \{h \in A: f_A(x) = f(x - h)\}$. The elements of $P_{f,A}(x)$ are said to be elements of f-best approximation (f-b.a.) to x from the set A. Throughout this paper we will assume that $f_A(\cdot)$ is proper.

A sensitivity analysis was first conducted by Brosowski–Deutch–Nürnberger [3], who considered a family $\{A_t\}_{t \in T}$ of subsets of a normed space X parametrized by a topological space T and studied the continuity of $t \to P_{A_t}(x)$ (here $f(\cdot) = \|\cdot\|$). Recently Tsukada [28] addressed the same problem but with a nonparametrized method. Namely he allowed the sets $\{A_n\}_{n \geqslant 1}$ to converge to A in the K–M sense and then examined what happens to the sequence $\{P_{A_n}(x)\}_{n \geqslant 1}$. His study was limited to strictly convex, reflexive Banach spaces.

Our first result examines the behavior of $P_{f,A}(x)$ under variations of the set A. Assume that X is a Banach space.

THEOREM 3.1. *If $f\colon X \to \mathbb{R}$ is continuous and w-sequentially l.s.c. and*

$$\{A_n, A\}_{n \geqslant 1} \subseteq P_{fc}(X) \qquad \text{s.t. } A_n \xrightarrow{\text{K-M}} A,$$

then for all $x \in X$, w-$\overline{\lim}\, P_{f, A_n}(x) \subseteq P_{f, A}(x)$.

Proof. Let $h \in w\text{-}\overline{\lim}\, P_{f, A_n}(x)$. Then by definition we can find $h_k \in P_{f, A_{n_k}}(x)$ s.t. $h_k \to^{w} h$. Let $y \in A$ and let $y_k \in A_{n_k}$ s.t. $y_k \to^{s} y$. This is possible since $A_n \to^{\text{K-M}} A$. Then using the properties of $f(\cdot)$ we have

$$\begin{aligned} f(x-h_k) &\leqslant f(x-y_k) \\ &\Rightarrow \underline{\lim}\, f(x-h_k) \leqslant \lim f(x-y_k) \\ &\Rightarrow f(x-h) \leqslant f(x-y). \end{aligned}$$

Note that $h \in A$ and since $y \in A$ was arbitrary we conclude that $h \in P_{f, A}(x)$. Q.E.D.

Remark. Note that with our assumptions on $f(\cdot)$ we cover the case where $f(\cdot) = \|\cdot\|$. We could have also assumed that $f(\cdot)$ is w-sequentially continuous.

Let $f(\cdot) = \|\cdot\|$. Recall that if X is reflexive and strictly convex then every $A \in P_{fc}(X)$ is a Chebyshev set, i.e., $P_A(x)$ is a singleton for all $x \in X$. Also X is said to have property (H) if and only if for every $x_n \to^{w} x$ with $\|x_n\| \to \|x\|$ we have $x_n \to^{s} x$. Locally uniformly convex spaces (in particular Hilbert spaces) have property (H). Using Theorem 3.1 we can have the following corollary which is Theorem 3.2(i) of Tsukada [28].

COROLLARY (28). *If X is reflexive and strictly convex and $\{A_n, A\}_{n \geqslant 1} \subseteq P_{fc}(X)$ with*

$$A_n \xrightarrow{\text{K-M}} A \qquad \text{as} \quad n \to \infty,$$

then for all $x \in X$, $P_{A_n}(x) \to^{w} P_A(x)$ as $n \to \infty$.
If in addition X has property (H) *then $P_{A_n}(x) \to^{s} P_A(x)$.*

Proof. Since $P_{A_n}(x)$, $P_A(x)\; n \geqslant 1$ are singletons, the first claim of the corollary follows directly from Theorem 3.1. Then $\|x - P_A(x)\| \leqslant \underline{\lim}\, \|x - P_{A_n}(x)\|$. Let $y_n \in A_n$ s.t. $y_n \to^{s} P_A(x)$. Then $\|x - y_n\| \to \|x - P_A(x)\|$. Since $\|x - P_{A_n}(x)\| \leqslant \|x - y_n\|$ we get that $\overline{\lim}\, \|x - P_{A_n}(x)\| \leqslant \|x - P_A(x)\|$. Thus $\|x - P_{A_n}(x)\| \to \|x - P_A(x)\|$ and because X has property (H) we get that $P_{A_n}(x) \to^{s} P_A(x)$. Q.E.D.

Remark. In the above proof we also got that $d_{A_n}(\cdot) \to d_A(\cdot)$.

Next we will allow $f(\cdot)$ to vary too. Then we have the following variant of Theorem 3.1. Assume X if finite dimensional. By $\delta_A(\cdot)$ we will denote the indicator function of $A \subseteq X$.

THEOREM 3.2. *If* $\{f_n, f\}_{n \geq 1} \subseteq \mathbb{R}^X$ *are continuous, convex,* $f_n \to f$ *as* $n \to \infty$ *and* $\{A_n, A\}_{n \geq 1} \subseteq P_{fc}(X)$ *s.t.* $A_n \to^{\text{K-M}} A$ *then for all* $x \in X$, $\overline{\lim}\, P_{f_n, A_n}(x) \subseteq P_{f, A}(x)$.

Proof. For any $x \in X$, let $f_x \colon X \to \mathbb{R}$ be defined by $f_x(y) = f(x-y)$. Then from Corollary 2E of Salinetti–Wets [25] we have $f_{n,x}(\cdot) \to^{\tau} f_x(\cdot)$ as $n \to \infty$. Also since $A_n \to^{\text{K-M}} A$, we have that $\delta_{A_n}(\cdot) \to^{\tau} \delta_A(\cdot)$. Note that $\operatorname{dom} f_x - \operatorname{dom} \delta_A = \mathbb{R}^n - A = \mathbb{R}^n$. So by Theorem 5 of McLinden–Bergstrom [18] we get that $(f_{n,x} + \delta_{A_n})(\cdot) \to^{\tau} (f_x + \delta_A)(\cdot) \Rightarrow (f_{n,x} + \delta_{A_n})^*(\cdot) \to^{\tau^*} (f_x + \delta_A)^*(\cdot) \Rightarrow \operatorname{Gr}(f_{n,x} + \delta_{A_n})^* \to^{\text{K-M}} \operatorname{Gr} \partial(f_x + \delta_A)^*(\cdot)$. But recall that $P_{f_n, A_n}(x) = \partial[f_{n,x} + \delta_{An}]^*(0)$ and $P_{f,A}(x) = \partial[f_x + \delta_A]^*(0)$. Hence it follows easily that $\overline{\lim}\, P_{f_n, A_n}(x) \subseteq P_{f,A}(x)$. Q.E.D.

We will close this section, with a result analogous to Theorem 3.1 but for f-farthest points. If $f \in \overline{\mathbb{R}}^X$ is a proper function, $A \subseteq X$ is nonempty then we define $\hat{f}_A(x) = \sup_{y \in A} f(x-y)$ and $Q_{f,A}(x) = \{h \in A : \hat{f}_A(x) = f(x-h)\}$. To avoid trivialities we will always assume that $\hat{f}_A(\cdot)$ is proper. Let X be a Banach space. By $h(\cdot,\cdot)$ we will denote the Hausdorff distance on 2^X.

THEOREM 3.3. *If* $f \colon X \to \mathbb{R}$ *is continuous,* $\{A_n\}_{n \geq 1} \subseteq P_k(X)$ *and* $A_n \to^h A$ *as* $n \to \infty$ *then for all* $x \in X$, $\hat{f}_{A_n}(x) \to \hat{f}_A(x)$ *and* $s\text{-}\overline{\lim}\, Q_{f,A_n}(x) \subseteq Q_{f,A}(x)$.

Proof. First note that $A \in P_k(X)$. For any $y \in A$, Let $y_n \in A_n$ s.r. $y_n \to^s y$. Then we have $f(x-y_n) \leq \hat{f}_{A_n}(x) \Rightarrow f(x-y) \leq \underline{\lim}\, \hat{f}_{A_n}(x) \Rightarrow \hat{f}_A(x) \leq \underline{\lim}\, \hat{f}_{A_n}(x)$. Let $h_n \in A_n\, n \geq 1$ s.t. $\hat{f}_{A_n}(x) = f(x-h_n)$. Observe that $d_A(h_n) \leq d_{A_n}(h_n) + h(A_n, A) = h(A_n, A) \to 0$ as $n \to \infty$. So $d_A(h_n) \to 0$. Let $a_n \in A$ s.t. $d_A(h_n) = \|h_a - a_n\|$. By passing to a subsequence if necessary, we may assume that $a_n \to^s h \in A$. Then $\|h_n - h\| \leq \|h_n - a_n\| + \|a_n - h\| \to 0$. So $h_n \to^s h \in A$. This implies that

$$\overline{\lim}\, \hat{f}_{A_n}(x) = \lim f(x-h_n) = f(x-h) \leq \hat{f}_A(x).$$

Hence we finally have that $\hat{f}_{A_n}(\cdot) \to \hat{f}_A(\cdot)$. Next let $h_k \in Q_{f,A_{n_k}}(x)\ k \geq 1$ and $h_k \to^s h$. Then $\hat{f}_{A_{n_k}}(x) = f(x-h_k) \to f(x-h) = \hat{f}_A(x)$ as $k \to \infty \Rightarrow h \in Q_{f,A}(x)$ $\Rightarrow s\text{-}\overline{\lim}\, Q_{f,A_n}(x) \subseteq Q_{f,A}(x)$. Q.E.D.

4. STOCHASTIC APPROXIMATION

Throughout this section assume that (Ω, Σ, μ) is a complete probability space and X a separable Banach space.

In the first result on the f-approximation of random sets we examine how the pointwise approximations are related to the aggregate (integral) approximation. So we will obtain a relation between the functions

$$f_{F(\omega)}(x) = \inf\{f(x-y): y \in F(\omega)\}$$

and

$$f_{\int F}(x) = \inf\left\{f(x-y): y \in \int_\Omega F(\omega)\right\}.$$

THEOREM 4.1. *$F: \Omega \to P_f(X)$ is measurable with $S_F^1 \neq \phi$, $f: X \to \mathbb{R}$ is l.s.c., convex and for all $x(\cdot) \in \mathscr{L}^1(x) = \{x(\cdot) \in L_X^1(\Omega): x = \int_\Omega x(\omega)\, d\mu(\omega)\}$ and all $y(\cdot) \in S_F^1$, $f(x(\cdot) - y(\cdot))$ is quasintegrable and integrable for one such pair $(x'(\cdot), y'(\cdot))$ then $f_{\int F}(x) = [\oint_\Omega f_{F(\omega)}](x)$, where $\oint_\Omega$ denotes the operation of continuous infimal convolution.*

Proof. From Lemma 2.1 of Hiai–Umegaki [9] we know that $\omega \to f_{F(\omega)}(x)$ is measurable. So the continuous infimal convolution in the conclusion of the theorem is well defined. Also from the definition of the Aumann integral we have that

$$f_{\int F}(x) = \inf_{y \in \int F} f(x-y) = \inf_{y(\cdot) \in S_F^1} f\left(x - \int_\Omega y(\omega)\, d\mu(\omega)\right).$$

Let $x(\cdot) \in \mathscr{L}^1(X)$. Then we have

$$f_{\int F}(x) = \inf_{y(\cdot) \in S_F^1} f\left(\int_\Omega (x(\omega) - y(\omega))\, d\mu(\omega)\right).$$

Applying Jensen's inequality (see Kozek–Suchanecki [15, Corollary 7.1]) we get that

$$f_{\int F}(x) \leqslant \inf_{y(\cdot) \in S_F^1} \int_\Omega f(x(\omega) - y(\omega))\, d\mu(\omega).$$

Since $x(\cdot) \in \mathscr{L}^1(x)$ was arbitrary we have that

$$f_{\int F}(x) \leqslant \inf_{\substack{y(\cdot) \in S_F^1 \\ x(\cdot) \in \mathscr{L}^1(x)}} \int_\Omega f(x(\omega) - y(\omega))\, d\mu(\omega).$$

Applying Theorem 2.2 of [9] we have that

$$f_{\int F}(x) \leqslant \inf_{x(\cdot) \in \mathscr{L}^1(x)} \int_\Omega \inf_{y \in F(\omega)} f(x(\omega) - y)\, d\mu(\omega)$$

$$= \inf_{x(\cdot) \in \mathscr{L}^1(x)} \int_\Omega f_{F(w)}(x(\omega))\, d\mu) = \left[\oint_\Omega f_{F(\omega)} \right](x).$$

Hence we have shown that

$$f_{\int F}(x) \leqslant \left[\oint_\Omega f_{F(\omega)} \right](x). \tag{1}$$

Next let $h \in \int_\Omega F(\omega)\, d\mu(\omega)$ s.t. $f(x-h) \leqslant f_{\int F}(x) + \varepsilon$, $\varepsilon > 0$. Note that $h = \int_\Omega h(\omega)\, d\mu(\omega)$ with $h(\cdot) \in S_F^1$. Let $x(\cdot) = x - h + h(\cdot)$. Clearly $x(\cdot) \in \mathscr{L}^1(x)$ and $f(x(\omega) - h(\omega)) = f(x-h) \leqslant f_{\int F}(x) + \varepsilon$. Also $f_{F(\omega)}(x(\omega)) \leqslant f(x(\omega) - h(\omega))$. So we have $f_{F(\omega)}(x(\omega)) \leqslant f_{\int F}(x) + \varepsilon$. Integrating both sides over Ω we get that

$$\int_\Omega f_{F(\omega)}(x(\omega))\, d\mu(\omega) \leqslant f_{\int F}(x) + \varepsilon$$

$$\Rightarrow \inf_{x(\cdot) \in \mathscr{L}^1(x)} \int_\Omega f_{F(\omega)}(x(\omega))\, d\mu(\omega) \leqslant f_{\int F}(x) + \varepsilon.$$

$$\Rightarrow \left[\oint_\Omega f_{F(\omega)} \right](x) \leqslant f_{\int F}(x) + \varepsilon.$$

Let $\varepsilon \downarrow 0$. We have that

$$\left[\oint_\Omega f_{F(\omega)} \right](x) \leqslant f_{\int F}(x). \tag{2}$$

From (1) and (2) we conclude that $f_{\int F}(x) = [\oint_\Omega f_{F(\omega)}](x)$. Q.E.D.

Remark. If $f(\cdot)$ is such that $f(x(\cdot))$ is quasintegrable for all $x(\cdot) \in L_X^1(\Omega)$, then the theorem is true for all $x \in X$. This is the case if $f(\cdot)$ is bounded from below.

When $f(\cdot) = \|\cdot\|$, then the quasintegrability hypothesis is automatically satisfied and if $F\colon \Omega \to P_f(X)$ is as above then:

COROLLARY. *For all $x \in X$, $d_{\int F}(x) = [\oint_\Omega d_{F(\omega)}](x)$.*

For the pointwise approximation problem we can say more. By S_F we will denote the set of measurable selectors of $F(\cdot)$. Also a set $A \subseteq X$ nonempty is said to be f-proximinal if and only if for all $x \in X$, $P_{f,A}(x) \neq \varnothing$.

THEOREM 4.2. *If $f: X \to \bar{\mathbb{R}}$ is l.s.c. and $F: \Omega \to P_f(X)$ is measurable with $F(\omega)$ being f-proximinal for all $\omega \in \Omega$ then for all $x \in X$, there exists $h(\cdot) \in S_F$ s.t. for all $\omega \in \Omega$, $f_{F(\omega)}(x) = f(x - h(\omega))$.*

Proof. For $x \in X$ consider the multifunction $\omega \to P_{f, F(\omega)}(x)$. By definition we have that

$$P_{f, F(\omega)}(x) = \{h \in X: f(x-h) - f_{F(\omega)}(x) = g_x(\omega, h) = 0\} \cap F(\omega).$$

Recall that $\omega \to f_{F(\omega)}(x)$ is measurable, while $f(x - \cdot)$ is l.s.c. Hence $g_x(\omega, h)$ being the sum of two normal integrands is normal. Thus

$$\text{Gr } P_{f, F(\cdot)}(x) = \{(\omega, h): g_x(\omega, h) = 0\} \cap \text{Gr } F \in \Sigma \times B(X).$$

Apply Aumann's selection theorem to find $h: \Omega \to X$ measurable s.t. $h(\omega) \in P_{f, F(\omega)}(x)$ for all $\omega \in \Omega$. Therefore $f(x - h(\omega)) = f_{F(\omega)}(x)$. Q.E.D.

Remark. The above result is still true if instead of $x \in X$ we have a measurable function $x: \Omega \to X$.

Next consider the following integral functional

$$I_f(x(\cdot)) = \int_\Omega f(\omega, x(\omega))\, d\mu(\omega),$$

where $f: \Omega \times X \to \bar{\mathbb{R}}$ is a measurable integrand and $x: \Omega \to X$ is measurable. Additional hypotheses will be introduced later. Let $M \subseteq L_X^1(\Omega)$ and define

$$I_f^M(x(\cdot)) = \inf_{y(\cdot) \in M} I_f(x(\cdot) - y(\cdot)).$$

Having established this notation consider a multifunction $F: \Omega \to 2^X \setminus \{\phi\}$ s.t. $S_F^1 \neq \varnothing$. We will examine the following two problems.

(1) For $\omega \in \Omega$, find $h \in F(\omega)$ s.t. $f_{F(\omega)}(\omega, x(\omega)) = f(\omega, x(\omega) - h)$.

(2) Find $h(\cdot) \in S_F^1$ s.t. $I_f^{S_F^1}(x(\cdot)) = I_f(x(\cdot) - h(\cdot))$.

By $P_{f(\omega, \cdot), F(\omega)}(x(\omega))$, $\omega \in \Omega$ we will denote the solution set of problem (1), while by $P_{I_f, S_F^1}(x(\cdot))$ we will denote the solution set of (2).

Under normality and measurability hypotheses on $f(\cdot, \cdot)$ and $F(\cdot)$, respectively, we can show as in Theorem 4.2 that $\omega \to P_{f(\omega, \cdot), F(\omega)}(x(\omega))$ is nonempty, closed valued, and measurable.

Our next theorem compares those two solution sets.

THEOREM 4.3. *If $f: \Omega \times X \to \bar{\mathbb{R}}$ is a normal, convex integrand s.t. for all $x(\cdot) \in L_X^1(\Omega)$, $f(\cdot, x(\cdot))$ is integrable and $F: \Omega \to P_{wkc}(X)$ is integrably bounded, then for any $x(\cdot) \in L_X^1(\Omega)$ we have $P_{I_f, S_F^1}(x(\cdot)) = S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}$ and is a w-compact subset of $L_X^1(\Omega)$.*

Proof. First note that since for all $\omega \in \Omega, f(\omega, \cdot) \in \Gamma_0(x) = \{$proper, l.s.c., convex, $\bar{\mathbb{R}}$-valued functions defined on $X\}$ and $F(\omega) \in P_{wkc}(X)$, then by Weierstrass theorem we have that $P_{f(\omega, \cdot), F(\omega)}(x(\omega)) \neq \emptyset$. Furthermore as we already pointed out $\omega \to P_{f(\omega, \cdot), F(\omega)}(x(\omega))$ is a measurable multifunction, with w-compact, convex values. Also it is integrably bounded since $F(\cdot)$ is. So from Proposition 3.1 of [21] we deduce that $S^1_{P_{f(\cdot, \cdot), F(\cdot)}(x(\cdot))} \neq \emptyset$, is w-compact and convex in $L^1_X(\Omega)$.

It is clear that $S^1_{P_{f(\cdot, \cdot), F(\cdot)}(x(\cdot))} \subseteq P_{I_f, S^1_F}(x(\cdot))$. Let $h(\cdot) \in P_{I_f, S^1_F}(x(\cdot))$. Then $I_f^{S^1_F}(x(\cdot)) = I_f(x(\cdot) - h(\cdot))$. So we have

$$\int_\Omega [f(\omega, x(\omega) - h(\omega)) - f_{F(\omega)}(\omega, x(\omega))] \, d\mu(\omega) = 0.$$

Since $f(\omega, x(\omega)) - h(\omega)) \geqslant f_{F(\omega)}(\omega, x(\omega))$ μ-a.e. we get that

$$f(\omega, x(\omega)) - h(\omega)) = f_{F(\omega)}(\omega, x(\omega)) \ \mu\text{-a.e.}$$
$$\Rightarrow h(\cdot) \in S^1_{P_{f(\cdot, \cdot), F(\cdot)}(x(\cdot))}.$$

Thus the claim of the theorem follows. Q.E.D.

Remark. When $f(\cdot, \cdot) = \|\cdot\|$, then the above theorem tells us that for any $x(\cdot) \in L^1_X(\Omega)$, the best approximation from S^1_F is also a pointwise best approximation to $x(\omega)$ from $F(\omega)$ and vice versa. Furthermore the set of such best approximations is w-compact in $L^1_X(\Omega)$.

Working as above we can have an analogous result for f-farthest points.

THEOREM 4.4. *If $f: \Omega \times X \to \bar{\mathbb{R}}$ is $\Sigma \times B(X)$-measurable and w-u.s.c. in x and for all $x(\cdot) \in L^1_X(\Omega), f(\cdot, x(\cdot))$ is integrable, while $F: \Omega \to P_{wkc}(X)$ is integrably bounded, then $Q_{I_f, S^1_F}(x(\cdot)) = S^1_{Q_{f(\cdot, \cdot), F(\cdot)}(x(\cdot))}$ for all $x(\cdot) \in L^1_X(\Omega)$.*

Theorem 4.3 is useful in obtaining interesting information about the structure of certain f-proximinal sets.

If $f \in \bar{\mathbb{R}}^X$ is a proper function and A a nonempty subset of X, then we say that A is an f-sun if for each $x \in X \backslash A$ there exists $h \in P_{f,A}(x)$ s.t. $h \in P_{f,A}(h + \lambda(x - h))$ for all $\lambda > 0$. We will say that A is a strict f-sun if this is true for all $h \in P_{f,A}(x)$. Note that if $f(\cdot) = \|\cdot\|$, then the above definition reduces to the classical definition of solarity of a set (see Vlasov [29]).

As before X is a separable Banach space.

THEOREM 4.5. *If $f: \Omega \times X \to \mathbb{R}$ is a Caratheodory, sublinear integrand s.t. for all $x(\cdot) \in L^1_X(\Omega), f(\cdot, x(\cdot))$ is integrable and $F: \Omega \to P_{wkc}(X)$ is integrably bounded, then $F(\omega)$ is an $f(\omega, \cdot)$-sun μ-a.e. if and only if S^1_F is an I_f-sun.*

Proof. First assume that S_F^1 is a I_f-sun. This means that for all $x(\cdot) \in L_X^1(\Omega)$ there exists $h(\cdot) \in P_{I_f, S_F^1}(x(\cdot))$ s.t. $h(\cdot) \in P_{I_f, S_F^1}(h(\cdot) + \lambda(x(\cdot) - h(\cdot)))$ for all $\lambda > 0$. Let $x(\omega) \equiv x$ and use Theorem 4.3, to get that

$$h(\omega) \in P_{f(\omega, \cdot), F(\omega)}(x) \ \mu\text{-a.e.} \Rightarrow h(\omega) \in P_{f(\omega, \cdot), F(\omega)}(h(\omega) + \lambda(x - h(\omega))) \ \mu\text{-a.e.}$$

for all $\lambda > 0$, which means that $P_{f(\omega, \cdot), F(\omega)}(x)$ is an $f(\omega, \cdot)$ sun μ-a.e.

Now assume that $F(\omega)$ is an $f(\omega, \cdot)$-sun μ-a.e. By definition this means that for μ-almost all $\omega \in \Omega$ we have that, for all $x \in X$ there exists $h \in P_{f(\omega, \cdot), F(\omega)}(x)$ s.t. $h \in P_{f(\omega, \cdot), F(\omega)}(h + \lambda(x - h))$ for all $\lambda > 0$. Let $x(\cdot) \in L_X^1(\Omega)$ and consider the multifunction $\Gamma(\cdot)$ defined by

$$\Gamma(\omega) = \{h \in P_{f(\omega, \cdot), F(\omega)}(x(\omega)) \colon h \in P_{f(\omega, \cdot), F(\omega)}(h + \lambda(x(\omega) - h)), \lambda \geqslant 0\}.$$

From Govindarajulu–Pai [7] we know that $x \to P_{f(\omega, \cdot), F(\omega,)}(x)$ is u.s.c. So we can write that

$$\Gamma(\omega) = \bigcap_{\substack{\lambda \geqslant 0 \\ \lambda = \text{rational}}} P_{f(\omega, \cdot), F(\omega)}(h + \lambda(x(\omega) - h)).$$

$\Rightarrow \Gamma(\cdot)$ has a measurable graph (see [4, Theorem III-40]).

Applying Aumann's selection theorem to find $h \colon \Omega \to X$ measurable s.t. $h(\omega) \in \Gamma(\omega)$ for all $\omega \in \Omega$. So we have that

$$\begin{aligned} &h(\omega) \in P_{f(\omega, \cdot), F(\omega)}(h(\omega) + \lambda(x(\omega) - h(\omega))), \qquad \omega \in \Omega \\ &\Rightarrow h(\cdot) \in S^1_{f(\cdot, \cdot), F(\cdot)}(h(\cdot) + \lambda(x(\cdot) - h(\cdot))) \\ &\Rightarrow h(\cdot) \in P_{I_f, S_F^1}(h(\cdot) + \lambda(x(\cdot) - h(\cdot))), \qquad \lambda \geqslant 0. \end{aligned}$$

which proves that S_F^1 is an I_f-sun. Q.E.D.

Now we will pass to the integral functional $F \colon X \to \bar{\mathbb{R}}$ defined by $F(x) = \int_\Omega f(\omega, x)\, d\mu(\omega)$, where $f(\cdot, \cdot)$ is a convex integrand. The next result provides a necessary and sufficient condition for the existence of f-b.a.

Assume that X is a reflexive, separable Banach space.

PROPOSITION 4.1. *If $f \colon \Omega \times X \to \bar{\mathbb{R}}$ is a normal, convex integrand s.t. for all $x \colon \Omega \to X$ measurable and bounded, $f(\cdot, x(\cdot))$ is integrable and A is a non-empty, closed, convex subset of X, then for all $x \in X$ we have, $h \in P_{F, A}(x)$ if and only if there exists*

$$x^*(\cdot) \in S^1_{\partial f(\cdot, x - h)} \quad \text{s.t. } x^* = \int_\Omega x^*(\omega)\, d\mu(\omega) \quad \text{and} \quad \sigma_A(x^*) = (x^*, h).$$

Proof. Note that $\inf_{y\in A} F(x-y)=\inf_{y\in X}[F_x+\delta_A](y)$, where $F_x(y)=F(x-y)$. From convex analysis (see [24]) we know that $h\in P_{f,A}(x)$ if and only if $0\in\partial(F_x+\delta_A)(h)$. But from Theorem 23(a) of Rockafellar [24] we know that $F(\cdot)$ is continuous, convex. So applying the Moreau–Rockafellar theorem (see Laurent [17]) we get that

$$0\in\partial(F_x+\delta_A)(h)=\partial F_x(h)+\partial\delta_A(h).$$

Observe that $\partial F_x(h)=-\partial F(x-h)$. Also from Theorem 23 (b) of Rockafellar [24] we know that

$$\partial F(x-h)=\int_\Omega \partial f(\omega, x-h)\,d\mu(\omega).$$

So there exits $x^*(\cdot)\in S^1_{\partial f(\cdot,x-h)}$ s.t. $x^*=\int_\Omega x^*(\omega)\,d\mu(\omega)$ and $x^*\in\partial\delta_A(h)$. Hence $(x^*, y-h)\leqslant 0$ for all $y\in A$. Thus $\sigma_A(x^*)=(x^*, h)$. Q.E.D.

Useful for the purposes of numerical analysis is the concept of ε-f-best approximation. We will say that $h\in A$ is an ε-f-b.a. to x from A if we have $f(x-h)\leqslant f_A(x)+\varepsilon$. We will denote the set of ε-f-best approximations to A by $P^\varepsilon_{f,A}$. For those points we have a result analogous to Proposition 4.1.

Assume that X is a finite dimensional Banach space.

PROPOSITION 4.2. *If $f:\Omega\times X\to\mathbb{R}$ is a Caratheodory, convex integrand s.t. for all $x:\Omega\to X$ measurable and bounded $f(\cdot, x(\cdot))$ is integrable and $A\subseteq X$ is nonempty, closed, and convex, then $h\in P^\varepsilon_{F,A}(x)$ if and only if there exists $\varepsilon:\Omega\to\mathbb{R}_+$ measurable, $\varepsilon'\geqslant 0$, $x^*:\Omega\to X^*$ measurable s.t. $x^*(\omega)\in\partial f(\omega, x-h)$ μ-a.e.,*

$$x^*=\int_\Omega x^*(\omega)\,d\mu(\omega),\qquad \sigma_A(x^*)-\varepsilon'\leqslant(x^*, h)$$

and

$$\int_\Omega^{\varepsilon(\omega)}\varepsilon(\omega)\,d\mu(\omega)+\varepsilon'=\varepsilon.$$

Proof. Note that

$$P^\varepsilon_{F,A}(x)=\{h\in A: F(x-h)\leqslant F_A(x)+\varepsilon\}.$$

So $h\in P^\varepsilon_{F,A}(x)$ is equivalent to saying that for all $y\in A$ we have $F(x-h)\leqslant F(x-y)+\varepsilon \Leftrightarrow -\varepsilon\leqslant F_x(y)-F_x(h)\Leftrightarrow -\varepsilon\leqslant(F_x+\delta_A)(y)-(F_x+\delta_A)(h)\Leftrightarrow 0\in\partial_\varepsilon(F_x+\delta_A)(h)$. From Theorem 23(a) of Rockafellar

[24] we know that $F_x(\cdot)$ is continuous, convex on X. Also from Hiriart-Urruty [12] [13] we have

$$\partial_\varepsilon(F_x+\delta_A)(h)=\bigcup_{\substack{\varepsilon_1\geqslant 0\varepsilon_2\geqslant 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}}\{\partial_{\varepsilon_1}F_x(h)+\partial_{\varepsilon_2}\delta_A(h)\}.$$

Moreover, again from [13] we have that

$$\partial_{\varepsilon_1}F_x(h)=\bigcup_{\substack{\varepsilon(\cdot)\in\mathscr{L}^1(\varepsilon)\\ \varepsilon(\omega)\geqslant 0}}\int_\Omega \partial_{\varepsilon(\omega)}f_x(\omega,h)\,d\mu(\omega).$$

So $0\in\partial_\varepsilon(F_x+\delta_A)(h)$ is equivalent to saying that there exists $z^*\in\int_\Omega\partial_{\varepsilon(\omega)}f_x(\omega,h)\,d\mu(\omega)$ for which $-z^*\in\partial_{\varepsilon'}\delta_A(h)$ and $\int_\Omega\varepsilon(\omega)\,d\mu(\omega)+\varepsilon'=\varepsilon$, where $\varepsilon(\omega)\geqslant 0$. Hence $z^*=\int_\Omega z^*(\omega)\,d\mu(\omega)$ with $z^*(\omega)\in\partial_{\varepsilon(\omega)}f_x(\omega,h)$ μ-a.e. But note that $\partial_{\varepsilon(\omega)}f_x(\omega,h)=-\partial_{\varepsilon(\omega)}f(\omega,x-h)$. So finally we can find $x^*\colon\Omega\to X^*$ measurable s.t. $x^*(\omega)\in\partial_{\varepsilon(\omega)}f(\omega,x-h)$ μ-a.e., $x^*=\int_\Omega x^*(\omega)\,d\mu(\omega)$ and $x^*\in\partial_{\varepsilon'}\delta_A(h)$ which is equivalent to saying that $\sigma_A(x^*)-\varepsilon'\leqslant(x^*,h)$, with $\int_\Omega\varepsilon(\omega)\,d\mu(\omega)+\varepsilon'=\varepsilon$ and $\varepsilon(\omega)\geqslant 0$ for all $\omega\in\Omega$. Q.E.D.

Next we will have a result analogous to Proposition 4.1 for the integral functional $I_f(\cdot)$.

Assume X is a reflexive, separable Banach space.

PROPOSITION 4.3. *If $f\colon\Omega\times X\to\bar{\mathbb{R}}$ is a normal, convex, integrand s.t. for all $x(\cdot)\in L_X^1(\Omega)$, $I_f(x(\cdot))$ is finite and $F\colon\Omega\to P_{fc}(X)$ is integrably bounded, then for any $x(\cdot)\in L_X^1(\Omega)$ we have $h(\cdot)\in P_{I_f,S_F^1}(x(\cdot))$ if and only if*

$$\min_{x^*(\cdot)\in S^\infty_{\operatorname{ext}\partial f(\cdot,x(\cdot)-h(\cdot))}}\int_\Omega(x^*(\omega),y(\omega)-h(\omega))\,d\mu(\omega)\leqslant 0\qquad\textit{for all}\quad y(\cdot)\in S_F^1.$$

Proof. As before $h(\cdot)\in P_{I_f,S_F^p}(x(\cdot))$ if and only if $0\in\partial(I_{f_{x(\cdot)}}+\partial_{S_F^1})(h(\cdot))$.

Recall that $I_{f_{x(\cdot)}}(\cdot)$ is continuous, convex on $L_X^1(\Omega)$. So we can apply the Moreau-Rockafellar theorem and get that $0\in\partial I_{f_{x(\cdot)}}(h(\cdot))+\partial\delta_{S_F^1}(h(\cdot))$. This means that there exists $-\hat{x}^*(\cdot)\in\partial I_{f_{x(\cdot)}}(h(\cdot))$ s.t. $\hat{x}^*(\cdot)\in\partial\delta_{S_F^1}(h(\cdot))$. Since $\partial I_{f_{x(\cdot)}}(h(\cdot))=-\partial(I_f(x(\cdot)-h(\cdot))$ we have $\hat{x}^*(\cdot)\in\partial I_f(x(\cdot)-h(\cdot))$ and $(\hat{x}^*(\cdot),y(\cdot)-h(\cdot))\leqslant 0$ for all $y(\cdot)\in S_F^p$. Recalling that $\partial I_f(x(\cdot)-h(\cdot))$ is w^*-compact we can write that

$$\begin{aligned}&\min_{x^*(\cdot)\in\partial I_f(x(\cdot)-h(\cdot))}(x^*(\cdot),y(\cdot)-h(\cdot))\\ &\quad=\min_{x^*(\cdot)\in\operatorname{ext}\partial I_f(x(\cdot)-h(\cdot))}(x^*(\cdot),y(\cdot)-h(\cdot))\leqslant 0.\end{aligned}$$

From Theorem 2.1(c) of Rockafellar [24] we know that

$$\partial I_f(x(\cdot)-h(\cdot)) = S^{\infty}_{\partial f(\cdot,\, x(\cdot)-h(\cdot))}$$
$$\Rightarrow \operatorname{ext} \partial I_f(x(\cdot)-h(\cdot)) = \operatorname{ext} S^{\infty}_{\partial f(\cdot,\, x(\cdot)-h(\cdot))}$$

Then from Benamara [2] we get that

$$\operatorname{ext} \partial I_f(x(\cdot)-h(\cdot)) = S^{\infty}_{\operatorname{ext} \partial f(\cdot,\, x(\cdot)-h(\cdot))}.$$

Therefore we have that

$$\min_{x^*(\cdot) \in S^{\infty}_{\operatorname{ext} \partial f(\cdot, x(\cdot)-h(\cdot))}} \int_{\Omega} (x^*(\omega), y(\omega)-h(\omega))\, d\mu(\omega) \leqslant 0$$

for all $y(\cdot) \in S^1_F$. Q.E.D.

Remark. In the above result instead of $p=1$, $q=+\infty$ we could have used a pair p, $q \geqslant 1$ of conjugate exponents.

We will close our investigation of the multifunction $P_{I_f, S^1_F}(\cdot)$ with a theorem concerning its continuity properties.

Assume X is a reflexive, separable Banach space.

THEOREM 4.6. *If $f\colon \Omega \times X \to \mathbb{R}$ is a Caratheodory, convex integrand s.t. for all $x\colon \Omega \to X$ bounded and measurable, $f(\cdot, x(\cdot))$ is integrable and $F\colon \Omega \to P_{wkc}(X)$ is integrably bounded, then $P_{I_f, S^1_F}(\cdot)$ is u.s.c. from L^1_X into (L^1_X, w).*

Proof. We know that for all $x(\cdot) \in L^1_X(\Omega)$, $P_{I_f, S^1_F}(x(\cdot)) = S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}$. So for any $x^* \in X^*$ we have $\sigma_{P_{I_f}, S^1_F}(x^*) = \sigma_{S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}}$. Then by definition

$$\sigma_{S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}}(x^*) = \sup_{h(\cdot) \in S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}} (x^*, h(\cdot)),$$

where $(\cdot,\cdot)$ denotes the duality brackets between L^1_X and $L^\infty_{X^*}$. Thus we have that

$$\sigma_{S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}}(x^*) = \sup_{h(\cdot) \in S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}} \int_{\Omega} (x^*, h(\omega))\, d\mu(\omega)$$

$$= \int_{\Omega} \sup_{h \in P_{f(\omega,\cdot), F(\omega)}(x(\omega))} d\mu(\omega) = \int_{\Omega} \sigma_{P_{f(\omega,\cdot), F(\omega)}(x(\omega))}(x^*)\, d\mu(\omega). \quad (*)$$

Next we will show that $x(\cdot) \to \sigma_{P_{I_f, S^1_F}(x(\cdot))}(x^*)$ is u.s.c. on L^1_X. So let $x_n(\cdot) \to^{s-L^1_X} x(\cdot)$. Then by passing to a subsequence if necessary we may assume that $x_n(\omega) \to^s x(\omega)$ μ-a.e. Using Fatou's lemma and (*) we get that

$$\overline{\lim}\, \sigma_{P_{I_f, S_F^1}(x_n(\cdot))}(x^*) \leqslant \int_\Omega \overline{\lim}\, \sigma_{P_{f(\omega,\cdot), F(\omega)}(x_n(\omega))}(x^*)\, d\mu(\omega).$$

From Govinarajulu–Pai [7] (Proposition 2.4), we know that $x \to P_{f(\omega,\cdot), F(\omega)}(x)$ is w-u.s.c. So Proposition 2 (p. 122) of Aubin–Ekeland [1] tells us that $x \to \sigma_{P_{f(\omega,\cdot), F(\omega)}(x)}(x^*)$ is w-u.s.c. So we have

$$\overline{\lim_{n\to\infty}}\, \sigma_{P_{f(\omega\cdot), F(\omega)}(x_n(\cdot))}(x^*) \leqslant \int_\Omega \sigma_{P_{f(\omega,\cdot), f(\omega)}(x(\omega))}(x^*)$$

$$\Rightarrow \overline{\lim_{n\to\infty}}\, \sigma_{P_{I_f, S_F^1}(x_n(\cdot))}(x^*) \leqslant \int_\Omega \sigma_{P_{f(\omega,\cdot), F(\omega)}(x(\omega))}(x^*)\, d\mu(\omega)$$

$$= \sigma_{S^1_{P_{f(\cdot,\cdot), F(\cdot)}(x(\cdot))}}(x^*) = \sigma_{P_{I_f, S_F^1}(x(\cdot))}(x^*)$$

$$\Rightarrow x(\cdot) \to \sigma_{P_{I_f}, S_F^1(x(\cdot))}(x^*) \qquad \text{is} \quad \text{u.s.c.}$$

Since $P_{I_f, S_F^1}(x(\cdot))$ has w-compact, convex values in L_X^1, Theorem 10 (p. 128) of [1] tells us that $x(\cdot) \to P_{I_f, S_F^1}(x(\cdot))$ is u.s.c. from L_X^1 into (L_X^1, w).
Q.E.D.

Remark. If for all $\omega \in \Omega$, $f(\omega, x) = \|x\|$ and $F(\omega)$ is Chebyshev and if $x_n(\cdot) \to^{s-L_X^1} x(\cdot)$ and $h_n(\cdot)$ are the best approximations to $x_n(\cdot)$ from S_F^1, then $h_n(\cdot) \to^{w-L_X^1} h(\cdot) =$ best approximation to $x(\cdot)$ from S_F^1.

We will close our study on stochastic f-best approximations, by examining what happens when we consider the conditional expectation of the integrand $f(\cdot,\cdot)$ with respect to a sub-σ-field Σ_0 of Σ.

Assume X is finite dimensional.

THEOREM 4.7. *If $f: \Omega \times X \to \mathbb{R}$ is a Caratheodory,* inf-*compact integrand s.t. for all $x \in X$, $\int_\Omega f(\omega, x)\, d\mu(\omega) \leqslant +\infty$ and $f(\omega, x) \geqslant a(\omega)$ μ-a.e., where $a(\cdot)$ is integrable and if $A \subseteq X$ is nonempty, closed, bounded, and convex, then for all $x \in X$, $(E^{\Sigma_0} f)_A(\omega, x) = E^{\Sigma_0} f_A(\omega, x)$ μ-a.e.*

Proof. First recall that for all $\omega \in \Omega$, $f_A(\omega, \cdot) = [f(\omega, \cdot) \,\square\, \delta_A](\cdot)$ and since by hypothesis $f(\omega, \cdot)$ is inf-compact, using Proposition 6.5.5 of Laurent [17] we deduce that $f_A(\omega, \cdot)$ is proper. l.s.c., and convex. Also Proposition 2R of Rockafellar [23] tells us that $f_A(\cdot, \cdot)$ is $\Sigma \times B(X)$-

measurable. Furthermore it is μ-a.e. bounded from below by $a(\omega)$ which is integrable. So $f_A(\cdot, \cdot)$ is a quasintegrable integrand and so we can consider its conditional expectation with respect to Σ_0. Let $B \in \Sigma_0$, $x: \Omega \to X$ bounded, Σ_0-measurable and $y \in A$. We have

$$\int_B E^{\Sigma_0} f_A(\omega, x(\omega))\, d\mu(\omega) = \int_B f_A(\omega, x(\omega))\, d\mu(\omega) \leqslant \int_B f(\omega, x(\omega) - y)\, d\mu(\omega).$$

$$= \int_B E^{\Sigma_0} f(\omega, x(\omega) - y)\, d\mu(\omega).$$

Invoking Lemma 6 of Thibault [27] we get that

$$E^{\Sigma_0} f_A(\omega, x) \leqslant E^{\Sigma_0} f(\omega, x - y)$$

for all $\omega \in \Omega \backslash N_y$, $\mu(N_y) = 0$. Let $\{y_n\}_{n \geqslant 1}$ be dense in X. Then clearly $E^{\Sigma_0} f_A(\omega, x) \leqslant E^{\Sigma_0} f(\omega, x - y_n)$ μ-a.e. and so exploiting the continuity of $E^{\Sigma_0} f(\omega, \cdot)$ (see [27]) we have that $E^{\Sigma_0} f_A(\omega, x) \leqslant E^{\Sigma_0} f(\omega, x - y)$ μ-a.e. $\Rightarrow$ $E^{\Sigma_0} f_A(\omega, x) \leqslant (E^{\Sigma_0} f)_A(\omega, x)$ μ-a.e.

On the other hand for all $y: \Omega \to A$ Σ_0-measurable and all $x: \Omega \to X$, bounded, Σ_0-measurable and for $B \in \Sigma_0$, we have that

$$\int_B (E^{\Sigma_0} f)_A(\omega, x(\omega))\, d\mu(\omega) \leqslant \int_B E^{\Sigma_0} f(\omega, x(\omega) - y(\omega))\, d\mu(\omega)$$

$$= \int_B f(\omega, x(\omega) - y(\omega))\, d\mu(\omega)$$

$$\Rightarrow \int_B (E^{\Sigma_0} f)_A(\omega, x(\omega))\, d\mu(\omega)$$

$$\leqslant \inf_{y(\cdot) \in S_A^1(\Sigma_0)} \int_B f(\omega, x(\omega) - y(\omega))\, d\mu(\omega)$$

$$= \int_B \inf_{y \in A} f(\omega, x(\omega) - y)\, d\mu(\omega)$$

$$= \int_B f_A(\omega, x(\omega))\, d\mu(\omega)$$

$$= \int_B E^{\Sigma_0} f_A(\omega, x(\omega))\, d\mu(\omega).$$

As before we get that $(E^{\Sigma_0} f)_A(\omega, x) \leqslant E^{\Sigma_0} f_A(\omega, x)$ μ-a.e. (exceptional set is independent of x)

$$\Rightarrow (E^{\Sigma_0} f)_A(\omega, x) = E^{\Sigma_0} f_A(\omega, x) \ \mu\text{-a.e.} \qquad \text{Q.E.D.}$$

Now we change direction and pass to the study of stochastic f-farthest points. The first result shows that $S_{I_f, S_F^1}(x(\cdot))$ has a solarity type property. Assume X is any separable Banach space.

PROPOSITION 4.4. *If $f: \Omega \times X \to \mathbb{R}$ is a Caratheodory, sublinear integrand, $F: \Omega \to P_f(X)$ is a measurable multifunction with $S_F^1 \neq \emptyset$ and there is a $y_0(\cdot) \in S_F^1$ s.t. $I_f(y_0(\cdot)) < \infty$, then $h(\cdot) \in Q_{I_f, S_F^1}(x(\cdot))$ implies that $h(\cdot) \in Q_{I_f, S_F^1}(h(\cdot) + \lambda(x(\cdot) - h(\cdot)))$ for all $\lambda > 0$.*

Proof. By definition $h(\cdot) \in Q_{I_f, S_F^1}(x(\cdot))$ means that

$$\hat{I}_f^{S_F^1}(x(\cdot)) = I_f(x(\cdot) - h(\cdot)) = \int_\Omega f(\omega, x(\omega) - h(\omega))\, d\mu(\omega).$$

Also

$$\begin{aligned} \hat{I}_f^{S_F^1}(x(\cdot)) &= \sup_{y(\cdot) \in S_F^1} I_f(x(\cdot) - y(\cdot)) \\ &= \sup_{y(\cdot) \in S_F^1} \int_\Omega f(\omega, x(\omega) - y(\omega))\, d\mu(\omega) \\ &= \int_\Omega \sup_{y \in F(\omega)} f(\omega, x(\omega) - y)\, d\mu(\omega). \end{aligned}$$

So we have that

$$\int_\Omega \sup_{y \in F(\omega)} f(\omega, x(\omega) - y)\, d\mu(\omega) = \int_\Omega f(\omega, x(\omega) - h(\omega))\, d\mu(\omega).$$

Since $h(\omega) \in F(\omega)$ μ-a.e. we deduce that

$$\begin{aligned} f(\omega, x(\omega) - h(\omega)) &= \sup_{y \in F(\omega)} f(\omega, x(\omega) - y) \ \mu\text{-a.e.} \\ &\Rightarrow h(\omega) \in Q_{f(\omega, \cdot), F(\omega)}(x(\omega)) \ \mu\text{-a.e.} \end{aligned}$$

From Proposition 2.8 of Govindarajulu–Pai [8] we know that for all $\lambda > 0$, $h(\omega) \in Q_{f(\omega, \cdot), F(\omega)}(h(\omega) + \lambda(x(\omega) - h(\omega)))$ μ-a.e. So for all $y(\cdot) \in S_F^1$ we have

$$\begin{aligned} & f(\omega, h(\omega) + \lambda(x(\omega) - h(\omega)) - y(\omega)) \leqslant f(\omega, \lambda(x(\omega) - h(\omega))) \ \mu\text{-a.e.} \\ &\Rightarrow I_f(h(\cdot) + \lambda(x(\cdot) - h(\cdot)) - y(\cdot)) \leqslant I_f(\lambda(x(\cdot) - h(\cdot))) \\ &\Rightarrow h(\cdot) \in Q_{I_f, S_F^1}(h(\cdot) + \lambda(x(\cdot) - h(\cdot))) \text{ for all } \lambda > 0. \end{aligned}$$

Q.E.D.

The next theorem provides a pointwise necessary condition for $h(\cdot)$ to be in $Q_{I_f, S_F^1}(x(\cdot)))$.

Assume as before that X is a separable Banach space. If $A \subseteq X$, by $N_A(x)$ we will denote the normal cone to A at the point x (see Clarke [5]).

THEOREM 4.8. *If* $f: \Omega \times X \to \mathbb{R}$ *is a* $k(\cdot)$*-Lipschitz,* L^1*-measurable integrand with* $k(\cdot) \in L^\infty(\Omega)$ *and* $F: \Omega \to P_{wkc}(X)$ *is integrably bounded, then* $h(\cdot) \in Q(x(\cdot))$ *implies that* $(-\partial f(\omega, x(\omega) - h(\omega))) \cap N_{F(\omega)}(h(\omega)) = \phi$ μ*-a.e.*

Proof. We will start by showing that if $h(\cdot) \in Q_{I_f, S_F^1}(x(\cdot)_1)$, then for some $l > 0$, $h(\cdot)$ also solves locally the following maximization problem

$$\sup_{z(\cdot) \in L_X^1} [I_{f_{x(\cdot)}} - l d_{S_F^1}](z(\cdot)). \tag{$*$}$$

Suppose not. Then for every $n \geqslant 1$ there exists $z_n(\cdot) \to^{s - L_X^1} h(\cdot)$ s.t.

$$I_{f_{x(\cdot)}}(z_n(\cdot)) - n d_{S_F^1}(z_n(\cdot)) > I_f(x(\cdot) - h(\cdot)) = \hat{I}_f^{S_F^1}(x(\cdot)).$$

Then $I_{f_{x(\cdot)}}(z_n(\cdot)) - I_{f_{x(\cdot)}}(h(\cdot)) > n d_{S_F^1}(z_n(\cdot))$ and so $\beta_n = d_{S_F^1} z_n(\cdot)) \to 0$ as $n \to \infty$. Recall that S_F^1 is w-compact in L_X^1. So we can find $h_n(\cdot) \in S_F^1$ s.t. $d_{S_F^1}(z_n(\cdot)) = \beta_n = \|z_n - h_n\|_1$ for all $n \geqslant 1$. So we have

$$I_f(x(\cdot) - h_n(\cdot)) \leqslant I_f(x(\cdot) - h(\cdot)) = \hat{I}_f^{S_F^1}(x(\cdot)) < I_f(x(\cdot) - z_n(\cdot)) - n\beta_n. \tag{1}$$

On the other hand, from the Lipschitzness hypothesis we have that

$$\begin{aligned} I_f(x(\cdot) - z_n(\cdot)) &\leqslant I_f(x(\cdot) - h_n(\cdot)) + \int_\Omega k(\omega)\, \|z_n(\omega) - h_n(\omega)\|\, d\mu(\omega) \\ &\leqslant I_f(x(\cdot) - h_n(\cdot)) + \|k\|_\infty \beta_n. \end{aligned}$$

Let $n \geqslant 1$ be such that $\|k\|_\infty \leqslant n$. Then we have

$$I_f(x(\cdot) - z_n(\cdot)) - \eta\beta_n \leqslant I_f(x(\cdot) - h_n(\cdot)). \tag{2}$$

From (1) and (2) above we produce the desired contradiction. Hence, knowing that $h(\cdot)$ solves $(*)$ locally, we can write that

$$0 \in \partial[I_{f_{x(\cdot)}} - l d_{S_F^1}](h(\cdot)),$$

where the subdifferential here is the generalized subdifferential in the sense of Clarke [5]. Recall that Clarke's subdifferential is subadditive. So we have that

$$\begin{aligned}
&0 \in \partial I_{f_{x(\cdot)}}(h(\cdot)) + \partial[-l d_{S_F^1}](h(\cdot)) \\
&\quad \Rightarrow 0 \in \partial I_{f_{x(\cdot)}}(h(\cdot)) - l \cdot \partial d_{S_F^1}(h(\cdot)) = -\partial I_f(x(\cdot) - h(\cdot)) - l \cdot \partial d_{S_F^1}(h(\cdot)) \\
&\quad \Rightarrow -\partial I_f(x(\cdot) - h(\cdot)) \cap (l \partial d_{S_F^1}(h(\cdot))) \neq \varnothing.
\end{aligned}$$

From Clarke [6] we know that $\partial I_f(x(\cdot) - h(\cdot)) \subseteq S^1_{\partial f(\cdot,\, x(\cdot) - h(\cdot))}$. Also note that $d_{S_F^1}(\cdot)$ and $d_{F(\omega)}(\cdot)(\omega \in \Omega)$, are Lipschitz, convex functions. Moreover, it is easy to see that for any $v(\cdot) \in L^1_X(\Omega)$

$$d_{S_F^1}(v) \leqslant \int_\Omega d_{F(\omega)}(v(\omega))\, d\mu(\omega) = I_{d_F}(v(\cdot)).$$

Thus for $h(\cdot) \in S_F^1$, we have that

$$\partial d_{S_F^1}(h) \subseteq \partial I_{d_F}(h) = S^1_{\partial_{F(\cdot)} d(h(\cdot))}.$$

Combining all the above observations we deduce that there exists $x^*\colon \Omega \to X^*$ s.t.

$$-x^*(\omega) \in \partial f(\omega, x(\omega) - h(\omega))\ \mu\text{-a.e.} \qquad \text{and} \qquad x^*(\omega) \in l \partial d_{F(\omega)}(h(\omega))\ \mu\text{-a.e.}$$

But recall that $l \partial d_{F(\omega)}(h(\omega)) \subseteq N_{F(\omega)}(h(\omega))$ for all $\omega \in \Omega$. So $x^*(\omega) \in N_{F(\omega)}(h(\omega))$ μ-a.e. Therefore finally we have

$$(-\partial f(\omega, x(\omega) - h(\omega))) \cap N_{F(\omega)}(h(\omega)) \subseteq \phi\ \mu\text{-a.e.} \qquad \text{Q.E.D.}$$

Now we turn for a while our attention to the pointwise maximization problem and we will examine the multifunction $\omega \to Q_{f(\omega,\cdot),\, F(\omega)}(x(\omega))$.

As always X is a separable Banach space.

PROPOSITION 4.5. *If $f\colon \Omega \times X \to \bar{\mathbb{R}}$ is a measurable integrand and $F\colon \Omega \to P_f(X)$ is a measurable multifunction then for all $x\colon \Omega \to X$ measurable, $\omega \to Q_{f(\omega,\cdot),\, F(\omega)}(x(\omega))$ is graph measurable.*

Proof. By definition we have that

$$\begin{aligned}
Q_{f(\omega,\cdot),\, F(\omega)}(x(\omega)) &= \{h \in F(\omega) : f(\omega, x(\omega) - h) = \hat{f}_{F(\omega)}(x(\omega))\} \\
&= \{h \in X : f(\omega, x(\omega) - h) \geqslant \hat{f}_{F(\omega)}(x(\omega))\} \cap F(\omega).
\end{aligned}$$

We claim that $\omega \to \hat{f}_{F(\omega)}(x(\omega))$ is measurable. To see that let $\lambda > 0$. Then we have that $\hat{f}_{F(\omega)}(x(\omega)) > \lambda$ if and only if there exists $y \in F(\omega)$ s.t.

$f(\omega, x(\omega)-y) > \lambda$. So we can write that $\{\omega \in \Omega: f(x(\omega) > \lambda\} = \mathrm{proj}_\Omega[\{(\omega, y) \in \Omega \times X: \quad f(\omega, x(\omega)-y) > \lambda\} \cap \mathrm{Gr}\, F]$. Recall that $\mathrm{Gr}\, F \in \Sigma \times B(X)$. Also since $f(\cdot, \cdot)$ is $\Sigma \times B(X)$-measurable, we have $\{(\omega, y) \in \Omega \times X: f(\omega, x(\omega)) - y > \lambda\} \in \Sigma \times B(X)$. So their intersection is in $\Sigma \times B(X)$. Then the projection theorem tells us that $\{\omega \in \Omega: \hat{f}_{F(\omega)}(x(\omega)) > \lambda\} \in \Sigma$. Hence $\omega \to \hat{f}_{F(\omega)}(x(\omega))$ is measurable. From this we deduce that $(\omega, h) \to \phi(\omega, h) = f(\omega, x(\omega)-h) - \hat{f}_{F(\omega)}(x(\omega))$ is $\Sigma \times B(X)$-measurable. Now observe that

$$\mathrm{Gr}\, Q_{f(\cdot,\cdot), F(\cdot)}(x(\cdot)) = \{(\omega, h) \in \Omega \times X: \phi(\omega, h) \geqslant 0\} \cap \mathrm{Gr}\, F \in \Sigma \times B(X).$$

Q.E.D.

An interesting consequence of this proposition is the following result. Assume that the same set of hypotheses is still in effect.

COROLLARY. *If for all* $\omega \in \Omega$, $Q_{f(\omega,\cdot), F(\omega)}(x(\omega)) \neq \phi$, *then there exists* $h: \Omega \to X$ *measurable selector of* $F(\cdot)$ *s.t.*

$$\hat{f}_{F(\omega)}(x(\omega)) = f(x(\omega) - h(\omega)).$$

Proof. From the previous proposition we know that $\mathrm{Gr}\, Q_{f(\cdot,\cdot), F(\cdot)}(x(\cdot)) \in \Sigma \times B(X)$. Apply Aumann's selection theorem to get $h: \Omega \to X$ measurable s.t. $h(\omega) \in F(\omega)$ and $\hat{f}_{F(\omega)}(x(\omega)) = f(\omega, x(\omega) - h(\omega))$ for all $\omega \in \Omega$. Q.E.D.

Under mild regularity assumptions on $f(\cdot, \cdot)$, we can have the following interesting characterization of the pointwise stochastic f-farthest points.

PROPOSITION 4.6. *If* $f: \Omega \times X \to \bar{\mathbb{R}}$ *is a measurable integrand s.t. for all* $\omega \in \Omega, f(\omega, \cdot)$ *is proper, convex, u.s.c., and* $F: \Omega \to P_{wkc}(X)$ *is measurable, then given any* $x \in X$, *we can find* $h: \Omega \to X$ *measurable s.t. for all* $\omega \in \Omega$, $h(\omega) \in \mathrm{ext}\, F(\omega)$ *and* $h(\omega) \in Q_{f(\omega,\cdot), F(\omega)}(x(\omega))$.

Proof. We saw in the proof of Proposition 4.5 that $\omega \to \hat{f}_{F(\omega)}(x)$ is measurable. Consider the multifunction $G(\cdot)$ defined by

$$G(\omega) = \{h \in \mathrm{ext}\, F(\omega): f(\omega, x-h) = \hat{f}_{F(\omega)}(x)\}.$$

From Bauer's maximum principle, we get that for all $\omega \in \Omega$, $G(\omega) \neq \varnothing$. Also $\mathrm{Gr}\, G = \{(\omega, h) \in \Omega \times X: f(\omega, x-h) - \hat{f}_{F(\omega)}(x) = 0\} \cap \mathrm{Gr}(\mathrm{ext}\, F)$. Recall that the first set in the intersection is in $\Sigma \times B(X)$, while from Benamara [2] we know that $\mathrm{Gr}(\mathrm{ext}\, F) \in \Sigma \times B(X)$. So we can apply Aumann's selection theorem to find $h: \Omega \to X$ measurable s.t. for all $\omega \in \Omega$, $h(\omega) \in G(\omega)$. This is the desired $h(\cdot)$. Q.E.D.

Remark. It is easy to check that this result holds true if instead of a fixed $x \in X$, we have $x: \Omega \to X$ measurable.

As with stochastic f-approximations, we will conclude our study of stochastic f-farthest points,by looking at the conditional expectation of $\hat{f}_A(\cdot)$ with respect to a sub-σ-field Σ_0 of Σ. The result is analogous to Theorem 4.7 but our assumptions on the space X and on the integrand $f(\cdot,\cdot)$ are less restrictive. The space X is as always a separable Banach space.

THEOREM 4.9. *If* $f: \Omega \times X \to \bar{\mathbb{R}}$ *is a normal integrand s.t. for all* $x(\cdot) \in L_X^1(\Omega)$, $f(\cdot, x(\cdot))$ *is integrable and there exists* $a(\cdot)$ *integrable s.t.* $f(\omega, x) \geqslant a(\omega)$ μ*-a.e. for all* $x \in X$ *and if* $F: \Omega \to P_f(X)$ *is* Σ_0*-measurable and integrably bounded then for all* $x(\cdot) \in L_X^1(\Sigma_0)$ *we have*

$$[E^{\hat{\Sigma}_0} f]_{F(\omega)}(\omega, x(\omega)) = E^{\Sigma_0} \hat{f}_{F(\omega)}(\omega, x(\omega)) \ \mu\text{-a.e.}$$

Proof. From Thibault [27] we know that there exists an increasing sequence of Caratheodory integrands $\{f_n(\cdot,\cdot)\}_{n \geqslant 1}$ s.t. for all $x \in X$ we have that

$$f(\omega, x) = \sup_{n \geqslant 1} f_n(\omega, x) \ \mu\text{-a.e.}$$

So we can write that

$$\begin{aligned}
\hat{f}_{F(\omega)}(\omega, x(\omega)) &= \sup_{y \in F(\omega)} f(\omega, x(\omega) - y) \\
&= \sup_{y \in F(\omega)} \sup_{n \geqslant 1} f_n(\omega, x(\omega) - y) \\
&= \sup_{n \geqslant 1} \sup_{y \in F(\omega)} f_n(\omega, x(\omega) - y) = \sup_{n \geqslant 1} (\hat{f}_n)_{F(\omega)}(\omega, x(\omega)) \ \mu\text{-a.e.}
\end{aligned}$$

Since X is separable and $f_n(\cdot,\cdot)$ are Caratheodory integrands, we have that $(\hat{f}_n)_{F(\cdot)}(\cdot,\cdot)$ are $\Sigma \times B(X)$-measurable. So $\hat{f}_{F(\cdot)}(\cdot,\cdot) = \sup_{n \geqslant 1} (\hat{f}_n)_{F(\cdot)}(\cdot,\cdot)$ is a normal integrand. Furthermore for all $x \in X$, $\hat{f}_{F(\omega)}(\omega, x) \geqslant a(\omega)$ μ-a.e. Hence $\hat{f}_{F(\cdot)}(\cdot,\cdot)$ is a quasintegrable, normal integrand and so $E^{\Sigma_0} \hat{f}(\cdot,\cdot)$ exists. Using Proposition 12 of Thibault [27] we have that for any $A \in \Sigma_0$ and any $y(\cdot) \in S_F^1(\Sigma_0)$,

$$\int_A E^{\Sigma_0} f(\omega, x(\omega) - y(\omega))\, d\mu(\omega) = \int_A f(\omega, x(\omega) - y(\omega))\, d\mu(\omega)$$

$$\Rightarrow \sup_{y(\cdot) \in S_F^1(\Sigma_0)} \int_A E^{\Sigma_0} f(\omega, x(\omega) - y(\omega))$$

$$= \sup_{y(\cdot) \in S^1_F(\Sigma_0)} \int_A f(\omega, x(\omega) - y(\omega))\, d\mu(\omega)$$

$$= \int_A \sup_{y \in F(\omega)} f(\omega, x(\omega) - y))\, d\mu(\omega)$$

$$= \int_A \hat{f}_{F(\omega)}(\omega, x(\omega))\, d\mu(\omega)$$

$$= \int_A E^{\Sigma_0} \hat{f}_{F(\omega)}(\omega, x(\omega))\, d\mu(\omega). \tag{1}$$

On the other hand note that

$$\sup_{y(\cdot) \in S^1_F(\Sigma_0)} \int_A E^{\Sigma_0} f(\omega), x(\omega) - y(\omega))\, d\mu(\omega)$$

$$= \int_A \sup_{y \in F(\omega)} E^{\Sigma_0} f(\omega, x(\omega) - y)\, d\mu(\omega)$$

$$= \int_A (E^{\hat{\Sigma}_0} f)(\omega, x(\omega))\, d\mu(\omega). \tag{2}$$

From (1) and (2) above we get that for all $A \in \Sigma_0$ and all $x(\cdot) \in L^1_X(\Sigma_0)$,

$$\int_A E^{\Sigma_0} \hat{f}_{F(\omega)}(\omega,\ x(\omega))\, d\mu(\omega) = \int_A (E^{\hat{\Sigma}_0} f\,)(\omega, x(\omega))\, d\mu(\omega).$$

Invoking Proposition 7 of [27] we conclude that

$$E^{\Sigma_0} \hat{f}_{F(\omega)}(\omega, x(\omega)) = (E^{\hat{\Sigma}_0} f)(\omega, x(\omega)) \ \mu\text{-a.e.} \qquad \text{Q.E.D.}$$

5. General Results

In this section we have gathered some useful general results about f-best approximations.

The first result illustrates how fixed point theory can be instrumental in obtaining interesting results about f-best approximations. Our theorem generalizes earlier results obtained by Ky Fan [16] and Reich [22].

Assume that X is a locally convex space. We recall that a set $A \subseteq X$ is said to be f-approximatively compact if and only if for all $x \in X$, every minimizing net $\{h_a\}$ (i.e., $f(x - h_a) \to f_A(x)$) has a convergent subnet in A.

THEOREM 5.1. *If $f: X \to \mathbb{R}$ is continuous, sublinear, $A \subseteq X$ is a nonempty f-approximatively compact, convex set, and ϕ: $A \to X$ is continuous with $\phi(A)$ compact, then there exists $h \in A$ s.t. $f_A(\phi(h)) = f(\phi(h) - h)$.*

Proof. Consider the multifunction $\Gamma: A \to P_{fc}(A)$ defined by $\Gamma(y) = (P_{f,A} \circ \phi)(y) = P_{f,A}(\phi(y))$. From [7, Proposition 2.4] we know that $P_{f,A}(\cdot)$ is u.s.c. while $\phi(\cdot)$ is by hypothesis cotinuous. So we deduce that $(P_{f,A} \circ \phi)(\cdot)$ is u.s.c. Moreover, we claim that $P_{f,A}(\cdot)$ has nonempty, compact, convex values. Nonemptiness follows from Proposition 2.1 of [7], while convexity is a straightforward consequence of the sublinearity of $f(\cdot)$. For compactness let $\{z_a\}$ be a net in $P_{f,A}(x)$. Then by definition we have that $f_A(x) = f(x - z_a)$. So $\{z_a\}$ is trivially a minimizing net in A. Because by hypothesis A is f-approximatively compact, we can find a subnet $\{z_b\}$ s.t. $z_b \to z \in A$. Also because of the continuity of $f(\cdot)$ we get that $f_A(x) = f(x - z)$. So $z \in P_{f,A}(x)$ and this proves that $P_{f,A}(\cdot)$ is compact valued. Then since $\phi(A)$ is compact, we have that $P_{f,A}(\phi(A))$ is compact. Applying Himmelberg's fixed point theorem [11], we get that there exists $h \in A$ s.t. $h \in \Gamma(h) = P_{f,A}(\phi(h)) \Rightarrow f_A(\phi(h)) = f(\phi(h) - h)$. Q.E.D.

Remark. When $f(\cdot) = p(\cdot)$, a continuous seminorm on X, then our theorem recovers the result of Reich [22].

We will conclude with two propositions on the properties of the multifunction $P_{f,A}(\cdot)$. In both X is assumed to be a locally convex space.

PROPOSITION 5.1. *If $f: X \to \mathbb{R}$ is continuous sublinear and $A \subseteq X$ is nonempty, f-approximatively compact, closed, and convex, then for any $K \subseteq X$ nonempty, connected, $P_{f,A}(K) = \bigcup_{x \in K} P_{f,A}(x)$ is connected too.*

Proof. For all $x \in X$, $P_{f,A}(x)$ is convex and so connected. Also recall that $P_{f,A}(\cdot)$ is u.s.c. Hence it maps connected sets to connected set. Thus $P_{f,A}(K)$ is connected. Q.E.D.

PROPOSITION 5.2. *If $f: X \to \mathbb{R}$ is continuous, sublinear and $A \subseteq X$ is nonempty closed, f-approximatively compact on $K \subseteq X$ nonempty and compact, then $P_{f,A}(K)$ is nonempty, compact and $\{x, h\} \in K \times A$: $h \in P_{f,A}(x)\}$ is compact in $X \times X$.*

Proof. Since A is f-approximatively compact on K, for all $x \in K$ $P_{f,A}(x) \in P_k(X)$. Also $P_{f,A}(\cdot)$ is u.s.c. on K. Then the claims of the proposition follow from the results of Smithson [26]. Q.E.D.

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